8823 – 7107M

Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

November 2023

Mathematics: analysis and approaches

Higher level

Paper 2

32 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme eg ***M1***, ***A2***.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***. If ***A1*** marks are on separate lines, they are assumed to be dependent and hence ***A0A1*** is unlikely to be awarded. However, where such marks are *independent* (e.g. the markscheme is presenting them in sequence, but in the solution one does not lead directly to the other) this should be communicated via a note, and hence ***A0A1*** (for example) can be awarded.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a “show that” question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal

approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part. Examples:

	Correct answer seen	**Further working seen**	**Any FT issues?**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	$0.468111\ldots$ (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version
- If the error leads to an inappropriate value (*e.g*. probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and $1{,}9$ or 1000 and 1,000 and 1.000.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

Final answers will generally not need to restate the variable and/or units to be considered correct. To help examiners, the markscheme will include variables and units, where appropriate. However, their omission from a candidate's final answer should only be penalized if explicitly instructed in a markscheme note.

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, , the general rule applies to final answers: unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to a "correct" level of accuracy (e.g 3 sf) in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

A GDC is required for this paper, but If you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

Section A

1. (a) $\text{BV} = \sqrt{(6-3)^2 + (8-4)^2 + (0-9)^2}$ ***(A1)***

$= 10.2956...$

$= 10.3\ (= \sqrt{106})$ ***A1***

Note: Award ***SC(A0)A1*** for $\text{BV} = \begin{pmatrix} -3 \\ -4 \\ 9 \end{pmatrix}$ where a candidate has misinterpreted notation.

[2 marks]

(b) **METHOD 1**

$\text{BV} = \text{VC}$ AND $\text{BC} = 8$ (seen anywhere) ***(A1)***

attempt to use the cosine rule on triangle BVC for any angle ***(M1)***

Note: Recognition must be shown in context either in terms of labelled sides or in side lengths.

$\cos \text{B}\hat{\text{V}}\text{C} = \dfrac{10.2...^2 + 10.2...^2 - 8^2}{2 \times 10.2... \times 10.2...}$ OR

$8^2 = 10.2....^2 + 10.2...^2 - 2 \times 10.2... \times 10.2... \cos \text{B}\hat{\text{V}}\text{C}$ ***(A1)***

$\text{B}\hat{\text{V}}\text{C} = 0.798037...$

$\text{B}\hat{\text{V}}\text{C} = 0.798$ (accept 45.7°) ***A1***

Note: If no working shown, award ***(A0)(M1)(A0)A0*** for $\text{B}\hat{\text{V}}\text{C} = 0.80$ (or 46°) (2sf).

METHOD 2

let M be the midpoint of BC

$\text{BM} = 4$ (seen anywhere) ***(A1)***

attempt to use sine or cosine in triangle BMV or CMV ***(M1)***

$\arcsin \dfrac{4}{\sqrt{106}}$ OR $\dfrac{\pi}{2} - \arccos \dfrac{4}{\sqrt{106}}$ OR 0.399018 ***(A1)***

$\text{B}\hat{\text{V}}\text{C} = 0.798037...$

$\text{B}\hat{\text{V}}\text{C} = 0.798$ (accept 45.7°) ***A1***

Note: If no working shown, award ***(A0)(M1)(A0)A0*** for $\text{B}\hat{\text{V}}\text{C} = 0.80$ (or 46°) (2sf).

METHOD 3

$\overrightarrow{\mathrm{VC}} = \begin{pmatrix} 3 \\ -4 \\ -9 \end{pmatrix}$ and $\overrightarrow{\mathrm{VB}} = \begin{pmatrix} 3 \\ 4 \\ -9 \end{pmatrix}$ ***(A1)***

attempt to use the cosine formula for an angle using two vectors representing the edges of triangle BVC ***(M1)***

$$\cos \mathrm{B\hat{V}C} = \frac{3(3) - 4(4) - 9(-9)}{\sqrt{3^2 + (-4)^2 + (-9)^2}\sqrt{3^2 + 4^2 + (-9)^2}} \left(= \frac{74}{106}\right)$$ ***(A1)***

$\mathrm{B\hat{V}C} = 0.798037...$

$\mathrm{B\hat{V}C} = 0.798$ (accept 45.7°) ***A1***

Note: If no working shown, award ***(A0)(M1)(A0)A0*** for $\mathrm{B\hat{V}C} = 0.80$ (or 46°) (2sf).

METHOD 4

$\overrightarrow{\mathrm{VC}} = \begin{pmatrix} 3 \\ -4 \\ -9 \end{pmatrix}$ and $\overrightarrow{\mathrm{VB}} = \begin{pmatrix} 3 \\ 4 \\ -9 \end{pmatrix}$ ***(A1)***

attempt to use the cross product formula for an angle using two vectors representing the edges of triangle BVC ***(M1)***

$$\sin \mathrm{B\hat{V}C} = \frac{\left|\begin{pmatrix} 72 \\ 0 \\ 24 \end{pmatrix}\right|}{\sqrt{3^2 + (-4)^2 + (-9)^2}\sqrt{3^2 + 4^2 + (-9)^2}} \left(= \frac{\sqrt{5760}}{106}\right)$$ ***(A1)***

$\mathrm{B\hat{V}C} = 0.798037...$

$\mathrm{B\hat{V}C} = 0.798$ (accept 45.7°) ***A1***

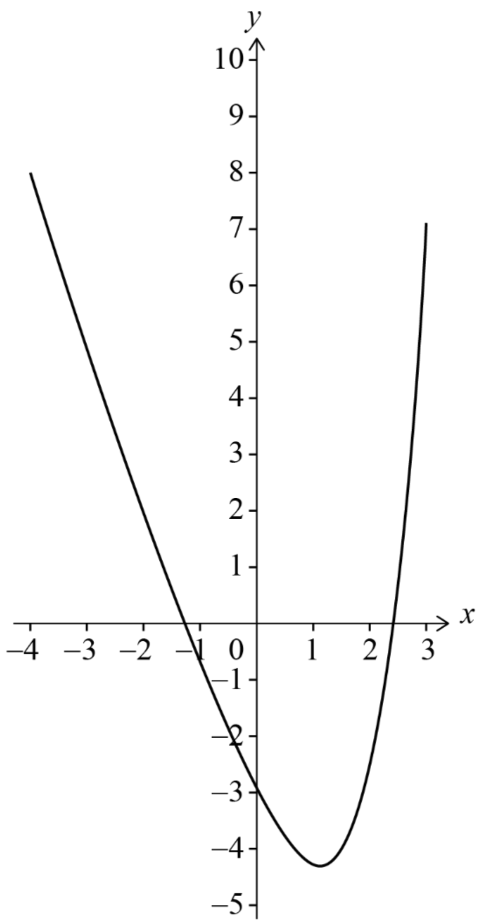
Note: If no working shown, award ***(A0)(M1)(A0)A0*** for $\mathrm{B\hat{V}C} = 0.80$ (or 46°) (2sf).

Award ***SC(A1)(M1)(A0)A0*** for $\text{area} = \frac{1}{2}\left|\begin{pmatrix} 72 \\ 0 \\ 24 \end{pmatrix}\right| = \frac{\sqrt{5760}}{2} (= 37.9)$ where a candidate has misinterpreted notation.

[4 marks]

Total [6 marks]

2. (a)



A1A1A1

Note:	Award marks as follows: ***A1*** for approximately correct roots, in the intervals $-2 < x < -1$ and $2 < x < 3$. ***A1*** for y-intercept AND local minimum in approximately correct positions. Allow for y-intercept $-3.5 < y < -2.5$, and for local minimum $0.5 < x < 1.5$, $-5 < y < -4$. ***A1*** for approximately correct endpoints, with the left end in the intervals $-4.5 < x < -3.5,\ 7.5 < y < 8.5$ and the right end in the intervals $2.5 < x < 3.5,\ 6.5 < y < 7.5$

[3 marks]

(b) $k = \frac{1}{2}$ ***A1***

$c = -3$ (accept translate/shift 3 (units) down) ***A1***

[2 marks]

Total [5 marks]

3. (a) use of sector area formula to find area of at least one sector ***(M1)***

$\frac{1}{2}\times 5.2\times 100-\frac{1}{2}\times 5.2\times r^2$ OR $10^2\pi-\frac{1}{2}10^2\times(2\pi-5.2)-\left(\pi r^2-\frac{1}{2}\times(2\pi-5.2)\times r^2\right)$ ***A1***

$(\text{area})=260-2.6r^2$ ***AG***

Note: There are many different ways to find the area of the "C". In all methods, the ***A*** mark is awarded for working which leads directly to the ***AG***.
Many candidates are working with rounded intermediate values. Award the ***A*** mark to correct work with values that round to the 3sf value of 260 and the 2sf value of 2.6 eg $259.99-2.6015r^2$.

[2 marks]

(b) (i) $260-2.6r^2=64$ ***(A1)***

$r=8.68243...$

$=8.68\,(\text{cm})\ \left(\frac{14\sqrt{65}}{13}\text{ exact}\right)$ ***A1***

(ii) 10×5.2 OR $8.68...\times 5.2$ ***(A1)***

substituting their value of r into $10\times 5.2+r\times 5.2+2(10-r)$ (or equivalent) ***(M1)***

Perimeter = $10\times 5.2+8.68...\times 5.2+2(10-8.68...)$ $(=52+45.1486...+2.63513...)$

= $99.7837\ldots$

= 99.8 (cm) ***A1***

[5 marks]

Total [7 marks]

4. (a) recognizing at rest when $\frac{\mathrm{d}s}{\mathrm{d}t} = 0$ OR s is a minimum ***(M1)***

$q = 5.73553...$

$= 5.74$ ***A1***

Note: If no working shown, award ***(M1)A0*** for $q = 5.7$ (2sf).

[2 marks]

(b) **METHOD 1**

recognizing that integral of $v(t)$ is required ***(M1)***

$\int_0^{5.73...} |v(t)|\,\mathrm{d}t$ OR $\int_0^{5.73...} \left|\frac{\mathrm{d}}{\mathrm{d}t}s(t)\right|\mathrm{d}t$ OR $\left|\int_0^{5.73...} v(t)\,\mathrm{d}t\right|$ OR $-\int_0^{5.73...} v(t)\,\mathrm{d}t$ ***(A1)***

Note: Condone absence of d*t*. Only accept $\left\lvert\int_0^q v(t)\,\mathrm{d}t\right\rvert$ if their value of q does not result in the particle changing direction in the first q seconds.

$= 7.68302...$

$= 7.68$ (m) ***A1***

Note: Special Cases: Award a maximum of ***(M1)(A1FT)A0FT*** if the candidate obtains $q = 1.62320...$ in part (a), and uses that value to find the total distance to be $3.38302...$ ($3.37644...$ from 3sf). Award ***(M1)(A0)A1*** if the candidate writes $\int_0^{5.73...} v(t)\,\mathrm{d}t$ followed by the correct answer.

METHOD 2

recognition that total distance travelled is the difference between the initial displacement and the displacement at minimum ***(M1)***

initial displacement is $3.38302...$ AND at minimum is -4.3 ***(A1)***

total distance travelled = $3.38302... - (-4.3)$

$= 7.68302...$

$= 7.68$ (m) ***A1***

Note: If no working shown, award ***(M1)(A0)A0*** for 7.7 (2sf).

[3 marks]

Total [5 marks]

5. $E(X) = k + 2k^2 + 3a + 4k^3 = 2.3$ ***(A1)***

$k + k^2 + a + k^3 = 1$ ***(A1)***

Note: The first two ***A*** marks are independent of each other.

EITHER (finding intersections of functions)

attempt to make a the subject in both of their equations ***(M1)***

$a = 1 - k - k^2 - k^3$ and $a = \frac{1}{3}\left(2.3 - k - 2k^2 - 4k^3\right)$

use of graph or table to attempt to find intersection ***(M1)***

OR (solving algebraically)

attempt to solve their equations algebraically to find a cubic in k ***(M1)***

$k^3 - k^2 - 2k + 0.7 = 0$ OR $3\left(1 - k - k^2 - k^3\right) = 2.3 - k - 2k^2 - 4k^3$ (or equivalent)

attempt to solve their cubic in k ***(M1)***

THEN

$a = 0.552839...$ OR $k = 0.315870...$ (other solutions to cubic are $k = -1.18538..., 1.86951...$)

$a = 0.553$ ***A1***

Note: If no working shown, award ***(A1)(A1)(M1)(M1)A0*** for $a = 2.44587...$ OR $a = -10.8987...$ and award ***(A0)(A0)(M1)(M1)A0*** for $a = 0.55$ (2sf) .

Total [5 marks]

6. (a) $5.75 = 25p(1-p)$ ***(A1)***

$p = 0.641421\ldots, 0.358578\ldots$

$p = 0.641, 0.359 \left(= \frac{5 \pm \sqrt{2}}{10}\right)$ ***A1A1***

[3 marks]

(b) $\text{Var}(Y) = (-2)^2 \text{Var}(X) (= 4\text{Var}(X))$ ***(A1)***

$= 23$ ***A1***

[2 marks]

Total [5 marks]

7. (a) (i) $(9! =)\ 362880$ ***A1***

Note: Accept 9! or 363000.

(ii) attempt to consider girls as a single object ***(M1)***

$(3! \times 7! =)\ 30240$ ***A1***

Note: Accept 30200.

[3 marks]

(b) **METHOD 1**

recognition of the two different cases for 2 girls and 3 girls ***(M1)***

exactly 2 girls is ${}^{6}C_{3} \times {}^{3}C_{2} = 60$ and exactly 3 girls $\left({}^{3}C_{3} \times\right) {}^{6}C_{2} = 15$ ***(A1)***

total $(= 60 + 15) = 75$ ***A1***

METHOD 2

recognition of the three different cases: total choices, 1 girl and no girls ***(M1)***

total choices ${}^{9}C_{5} = 126$, one girl case ${}^{3}C_{1} \times {}^{6}C_{4} = 45$, no girl case ${}^{6}C_{5} = 6$ ***(A1)***

total $(= 126 - 45 - 6) = 75$ ***A1***

[3 marks]

Total [6 marks]

8. (a) $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 1-p \\ -1 \end{pmatrix}$ ***A1***

$\overrightarrow{AC} = \begin{pmatrix} p \\ -p \\ 2 \end{pmatrix}$ ***A1***

attempt to evaluate their $\overrightarrow{AB} \times \overrightarrow{AC}$ by use of formula or determinant ***M1***

$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 2(1-p)-p \\ -(2+p) \\ -p-p(1-p) \end{pmatrix}$ OR $\left(2(1-p)-p\right)\boldsymbol{i} - (2+p)\,\boldsymbol{j} + \left(-p-p(1-p)\right)\boldsymbol{k}$ ***A1***

$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 2-3p \\ -2-p \\ p^2-2p \end{pmatrix}$ ***AG***

[4 marks]

(b) $\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|^2$

$= (2-3p)^2 + (-2-p)^2 + \left(p^2-2p\right)^2 \left(= p^4 - 4p^3 + 14p^2 - 8p + 8\right)$ ***(A1)***

attempt to find minimum of their $\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|^2$ ***(M1)***

$6.75257...$ OR $p = 0.3264...$

min value is 6.75 ***A1***

[3 marks]

continued…

Question 8 continued

(c) **METHOD 1**

valid attempt to find $\text{area} = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$ using their answer to part b) ***(M1)***

area $= \frac{1}{2}\sqrt{6.75257...}$

$= 1.299285...$

$= 1.30$ (units^2) ***A1***

[2 marks]

Total [9 marks]

9. (a) attempt to use recursive formula $y_n = y_{n-1} + 0.1\left(\frac{4 - y_{n-1}}{10}\right)$ ***(M1)***

n	x_n	y_n
0	0	2
1	0.1	2.02
2	0.2	2.0398
3	0.3	2.05940…
4	0.4	2.07880…
5	0.5	2.09801…

$y_1 = 2.02$ ***(A1)***

$y_5 = 2.098$ ***A1***

Note: Accept any answer which rounds to the correct 4sf value.

Award no marks for a final answer of 2.1 or 2.10 with no working.

[3 marks]

continued…

Question 9 continued

(b) **METHOD 1**

Note: Condone absence of absolute value signs throughout

$\int \frac{\mathrm{d}y}{4-y} = \int \frac{\mathrm{d}x}{10}$ ***M1***

$-\ln|4-y| = \frac{x}{10}(+c)$ ***A1***

EITHER

substituting initial conditions $x = 0, y = 2$ to find the value of c ***M1***

$(-\ln 2 = 0 + c \Rightarrow) c = -\ln 2$ ***A1***

$-\ln|4-y| = \frac{x}{10} - \ln 2 \Rightarrow \ln\frac{|4-y|}{2} = -\frac{x}{10}$

$|4-y| = 2\mathrm{e}^{-\frac{x}{10}}$ ***A1***

OR

$|4-y| = \mathrm{e}^{-\frac{x}{10}-c}$ (so $4-y = \pm\mathrm{e}^{-c}\mathrm{e}^{-\frac{x}{10}}$)

$4-y = A\mathrm{e}^{-\frac{x}{10}}$ ***A1***

substituting initial conditions $x = 0$, $y = 2$ to find the value of A ***M1***

$2 = A\mathrm{e}^0 \Rightarrow A = 2$ ***A1***

THEN

$y = 4 - 2\mathrm{e}^{-\frac{x}{10}}$ ***AG***

Note: Candidates may use $-\int \frac{\mathrm{d}y}{y-4} = \int \frac{\mathrm{d}x}{10}$ and correctly obtain $\lvert y-4\rvert = 2\mathrm{e}^{-\frac{x}{10}}$ leading to $4-y = 2\mathrm{e}^{-\frac{x}{10}}$ after consideration of the boundary conditions. In this case, the absence of absolute value signs should not be condoned until the sign has been resolved.

continued…

Question 9 continued

METHOD 2

attempt to rearrange and find an integrating factor ***M1***

$\frac{dy}{dx} + \frac{1}{10}y = \frac{4}{10}$ so IF $e^{\int \frac{1}{10}dx} = e^{\frac{1}{10}x}$

$e^{\frac{1}{10}x}\frac{dy}{dx} + \frac{1}{10}e^{\frac{1}{10}x}y = \frac{4}{10}e^{\frac{1}{10}x}$

$e^{\frac{1}{10}x}y = 4e^{\frac{1}{10}x}\left(+c\right)$ ***A1A1***

Note: Award ***A1*** for LHS and ***A1*** for RHS***.***

substituting initial conditions $x = 0, y = 2$ to find the value of c ***M1***

$\left(2e^0 = 4e^0 + c \Rightarrow\right) c = -2$ ***A1***

$e^{\frac{1}{10}x}y = 4e^{\frac{1}{10}x} - 2$

$y = 4 - 2e^{-\frac{x}{10}}$ ***AG***

[5 marks]

(c) absolute error $= 2.0980199... - \left(4 - 2e^{-0.05}\right) = 0.000478749...$

$= 0.000479\left(= 4.79\times10^{-4}\right)$ ***A1***

Note: Accept $0.000459\left(= 4.59\times10^{-4}\right)$ from use of 4sf value.

[1 mark]

Total [9 marks]

Section B

10. (a) recognizing probabilities sum to 1 ***(M1)***

$0.288 + \mathrm{P}(94.6 < X < 98.1) + 0.434 = 1$

$\mathrm{P}(94.6 < X < 98.1) = 0.278$ ***A1***

Note: If no working shown, award ***(M1)A0*** for $\mathrm{P}(94.6 < X < 98.1) = 0.28$ (2sf).

[2 marks]

(b) **METHOD 1**

recognizing the need to use inverse normal with 0.288, $(1-0.434)$ or 0.434 ***(M1)***

Note: Accept use of calculator notation eg $\mathrm{invNorm}\left(0.288\right)\left(= -\,0.559236...\right)$.

$\mu + \mathrm{invNorm}(0.288)\sigma = 94.6, \mu + \mathrm{invNorm}(1-0.434)\sigma = 98.1$ (or equivalent) ***(A1)(A1)***

attempt to solve their equations in two variables using the GDC (that involve either z-values or 'invNorm' rather than probabilities) ***(M1)***

$\mu = 97.2981..., \ \sigma = 4.82468...$

$\mu = 97.3, \ \sigma = 4.82$ ***A1***

Note: Condone use of different variables throughout, but do not award the final ***A1*** if they do not clearly identify which variable is their mean and standard deviation.

METHOD 2

use of inverse normal to find at least one z-score for $\mathrm{P}(Z < z) = 0.288$ or $\mathrm{P}(Z < z) = 1 - 0.434$ ***(M1)***

$z_1 = -0.559236...$ OR $z_2 = 0.166199...$

$\frac{94.6-\mu}{\sigma} = -0.559236...$, $\frac{98.1-\mu}{\sigma} = 0.166199...$ (or equivalent) ***(A1)(A1)***

attempt to solve their equations (that involve z-values rather than probabilities) ***(M1)***

$\mu = 97.2981..., \ \sigma = 4.82468...$

$\mu = 97.3, \ \sigma = 4.82$ ***A1***

Note: Award marks as appropriate for work seen in part (a).

Note: If no working shown, award ***(M1)(A0)(A0)(M1)A0*** for $\mu = 97, \ \sigma = 4.8$ (2sf).

[5 marks]

Question 10 continued

(c) (i) recognition of Binomial distribution ***(M1)***

$X \sim \mathrm{B}(100, 0.434)$

$\mathrm{P}(X = 34) = 0.0133198...$

$= 0.0133$ ***A1***

Note: If no working shown, award ***(M1)A0*** for $\mathrm{P}(X = 34) = 0.013$ (2sf).

(ii) $\mathrm{P}(X < 49) = 0.848218...$ (seen anywhere) ***(A1)***

recognition of conditional probability ***(M1)***

Note: recognition must be shown in context, either in symbols eg $\mathrm{P}\left(X = 34 \mid X < 49\right)$, or in words eg $\mathrm{P}\left(34 \text{ plants} \mid \text{less than 49 plants}\right)$, not only as $\mathrm{P}(A \mid B)$.

$\left(\mathrm{P}\left(X = 34 \mid X < 49\right) =\right) \dfrac{\mathrm{P}\left(X = 34\right)}{\mathrm{P}\left(X < 49\right)}$ OR $\dfrac{\mathrm{P}\left(X = 34\right)}{\mathrm{P}\left(X \le 48\right)}$ $\left(= \dfrac{0.0133198...}{0.848218...}\right)$ ***(A1)***

$= 0.0157033...$

$\mathrm{P}\left(X = 34 \mid X < 49\right) = 0.0157$ ***A1***

Note: Exception to ***FT***: If the candidate finds $\mathrm{P}(X \le 49)\left(= 0.890474...\right)$ and uses that to calculate $\mathrm{P}\left(X = 34 \mid X \le 49\right) = 0.0149581...$ award ***(A0)(M1)(A1)A0***.

Note: If no working shown, award ***(A0)(M1)(A0)A0*** for $\mathrm{P}\left(X = 34 \mid X < 49\right) = 0.016$ (2sf).

[6 marks]

continued…

Question 10 continued

(d) $Q_1 = 96.19$ OR $Q_3 = 101.01$ (may be seen on a labelled diagram with areas indicated) ***(A1)***

$P(96.19 < F < 101.01) = 0.5$ OR $P(F < 96.19) = 0.25$ OR $P(F < 101.01) = 0.75$ (or equivalent)

EITHER

attempt to find d using graph or table ***(M1)***

OR

$1 - 2P\left(Z < -\frac{2.41}{d}\right) = 0.5$ OR $P(Z < -\frac{2.41}{d}) = 0.25$ OR $P(Z < \frac{2.41}{d}) = 0.75$

OR $P(-\frac{2.41}{d} < Z < \frac{2.41}{d}) = 0.5$ (or equivalent) ***(M1)***

$-\frac{2.41}{d} = -0.674489...$ OR $\frac{2.41}{d} = 0.674489...$

THEN

$3.57307...$

$d = 3.57$ ***A1***

Note: Accept 3.56 using 96.2 or 101.

Note: If no working shown, award ***(A0)(M1)A0*** for $d = 3.6$ (2sf).

[3 marks]

Total [16 marks]

11. (a) (vertical asymptote equation) $x = -3$ ***A1***

Note: Accept $2x + 6 = 0$ or equivalent.

[1 mark]

(b) $(2, 0)$ and $(12, 0)$ ***A1A1***

Note: Award ***A1*** for $(2, 0)$ and ***A1*** for $(12, 0)$.
Award ***A1A0*** if only *x* values are given.

[2 marks]

(c) **METHOD 1**

$a = \frac{1}{2}$ ***A1***

attempt at 'long division' on $\frac{x^2 - 14x + 24}{2x + 6}$ ***(M1)***

$\frac{x^2 - 14x + 24}{2x + 6}$

$= \frac{1}{2}x - \frac{17}{2}\left(+\frac{...}{2x + 6}\right)$ ***(A1)***

$b = -\frac{17}{2}$ ***A1***

Note: Accept $y = \frac{1}{2}x - \frac{17}{2}$.

continued…

Question 11 continued

METHOD 2

$a=\frac{1}{2}$ ***A1***

$\frac{x^2-14x+24}{2x+6}\equiv\frac{1}{2}x+b+\frac{c}{2x+6}$ ***(A1)***

$x^2-14x+24\equiv\frac{1}{2}x(2x+6)+b(2x+6)+c$

attempt to equate coefficients of x: ***(M1)***

$-14=3+2b$

$b=-\frac{17}{2}$ ***A1***

Note: Accept $y=\frac{1}{2}x-\frac{17}{2}$.

METHOD 3

$a=\frac{1}{2}$ ***A1***

$\frac{x^2-14x+24}{2x+6}-\frac{1}{2}x\equiv\frac{-17x+24}{2x+6}$ ***(A1)***

attempt to find the limit of $f(x)-ax$ as $x\longrightarrow\infty$ ***(M1)***

$b=\lim_{x\to\infty}\frac{-17x+24}{2x+6}$

$=-\frac{17}{2}$ ***A1***

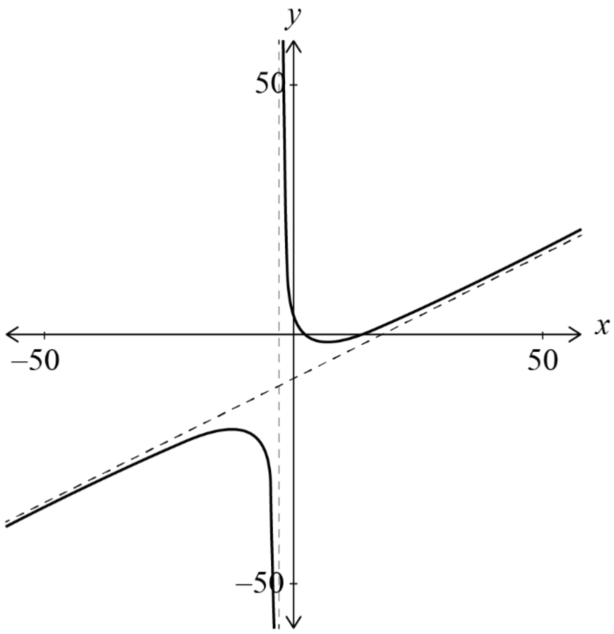
Note: Accept $y=\frac{1}{2}x-\frac{17}{2}$.

[4 marks]

continued…

Question 11 continued

(d)



two branches with approximately correct shape (for $-50 \le x \le 50$) ***A1***

Note: For this **A1** the graph must be a function.

their vertical and oblique asymptotes in approximately correct positions with both branches showing correct asymptotic behaviour to these asymptotes ***A1A1***

Note: Award ***A1*** for vertical asymptote and behaviour and ***A1*** for oblique asymptote and behaviour. If only top half of the graph seen only award ***A1A0*** if both asymptotes and behaviour are seen.

their axes intercepts in approximately the correct positions ***A1***

Note: Points of intersection with the axes and the equations of asymptotes do not need to be labelled. Ignore incorrect labels.

[4 marks]

(e) $\left(-10-5\sqrt{3}=\right)-18.6602....$ OR $\left(-10+5\sqrt{3}=\right)-1.33974....$ seen anywhere ***(A1)***

attempt to write the range using at least one value in an interval or an inequality in y or $f(x)$

(M1)

$y \le -18.7,\ y \ge -1.34$ ***A1A1***

Note: Award ***A1*** for each inequality**.** Award ***A1A0*** for strict inequalities in both. Do not award FT from (d). Accept equivalent set notation.

[4 marks]

(f) $\left(-10-2\sqrt{31}=\right)-21.1355....$ OR $\left(-10+2\sqrt{31}=\right)1.13552....$ seen anywhere ***(A1)***

$x < -21.1,\ -3 < x < 1.14$ ***A1A1A1***

Note: Award ***A1*** for $x < -21.1$, ***A1*** for correct endpoints of a single interval -3 and 1.14 and ***A1*** for $-3 < x < 1.14$. Do not award FT from (d). Accept equivalent set notation.

[4 marks]

Total [19 marks]

12. (a) attempt to set at least two components of L and M equal ***M1***

$1+2s=9+4t$

$2+3s=9+t$

$-3+6s=11+2t$

attempt to solve two of their equations simultaneously ***(M1)***

$s=2$ OR $t=-1$ ***A1***

EITHER

substitute $s=2$ and $t=-1$ into remaining component e.g. $-3+6(2)=11+2(-1)$ ***R1***

OR

recognition that 2nd and 3rd equations are equivalent ***R1***

THEN

position vector of A is $\begin{pmatrix}5\\8\\9\end{pmatrix}$ ***A1***

Note: Accept a row vector and/or coordinates. The final ***A1*** is independent of ***R1***.

[5 marks]

(b) **METHOD 1**

attempt to substitute at least one line into the equation of the plane ***(M1)***

$\begin{pmatrix}1+2s\\2+3s\\-3+6s\end{pmatrix}\bullet\begin{pmatrix}0\\2\\-1\end{pmatrix}=2(2+3s)-1(-3+6s)=7$ ***A1***

$\begin{pmatrix}9+4t\\9+t\\11+2t\end{pmatrix}\bullet\begin{pmatrix}0\\2\\-1\end{pmatrix}=2(9+t)-1(11+2t)=7$ ***A1***

continued…

Question 12 continued

METHOD 2

consideration the direction of one line and a point on that line ***(M1)***

direction $\begin{pmatrix}2\\3\\6\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=0$ AND point $\begin{pmatrix}1\\2\\-3\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=7$ $\left(\text{or } \begin{pmatrix}5\\8\\9\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=7\right)$ ***A1***

direction $\begin{pmatrix}4\\1\\2\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=0$ AND point $\begin{pmatrix}9\\9\\11\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=7$ $\left(\text{or } \begin{pmatrix}5\\8\\9\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=7\right)$ ***A1***

METHOD 3

consideration of direction of both lines ***(M1)***

EITHER

$\begin{pmatrix}2\\3\\6\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=0$ and $\begin{pmatrix}4\\1\\2\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=0$ (hence L and M are parallel to the plane) ***A1***

OR

$\begin{pmatrix}2\\3\\6\end{pmatrix}\times\begin{pmatrix}4\\1\\2\end{pmatrix}=\begin{pmatrix}0\\20\\-10\end{pmatrix}=k\begin{pmatrix}0\\2\\-1\end{pmatrix}$ (hence L and M are parallel to the plane) ***A1***

THEN

$\begin{pmatrix}5\\8\\9\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=7$ OR $\begin{pmatrix}1\\2\\-3\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=7$ OR $\begin{pmatrix}9\\9\\11\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=7$ ***A1***

[3 marks]

continued…

Question 12 continued

(c) (i) position vector of point on the line is $(\boldsymbol{r} =) \begin{pmatrix} -3 \\ 12 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \left(= \begin{pmatrix} -3 \\ 12+2\lambda \\ 2-\lambda \end{pmatrix} \right)$ ***(A1)***

attempt to substitute position vector into equation of plane Π ***(M1)***

meets Π when $\begin{pmatrix} -3 \\ 12+2\lambda \\ 2-\lambda \end{pmatrix} . \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = 7$

$2(12+2\lambda)-(2-\lambda)=7$

$22+5\lambda=7$

$\lambda=-3$ ***(A1)***

position vector of $\left(\boldsymbol{r} = \begin{pmatrix} -3 \\ 12 \\ 2 \end{pmatrix} - 3 \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = \right) \begin{pmatrix} -3 \\ 6 \\ 5 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \\ 5 \end{pmatrix}$ ***A1***

Note: Accept a row vector and/or coordinates.

(ii) **METHOD 1**

attempt to find $\overrightarrow{BC}$ using $\overrightarrow{OC}-\overrightarrow{OB}$ ***(M1)***

$\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=\begin{pmatrix} -3 \\ 6 \\ 5 \end{pmatrix} - \begin{pmatrix} -3 \\ 12 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -6 \\ 3 \end{pmatrix}$

attempt to use distance formula to find $\left|\overrightarrow{BC}\right|$ ***(M1)***

$\left|\overrightarrow{BC}\right| = \sqrt{(-6)^2+3^2}$

$=6.71\left(=\sqrt{45}=3\sqrt{5}\right)$ ***A1***

METHOD 2

recognition that $\left|\overrightarrow{BC}\right| = 3 \times \left|\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}\right|$ ***(M1)***

attempt to use distance formula to find $\left|\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}\right|$ ***(M1)***

$\left|\overrightarrow{BC}\right| = 3\sqrt{2^2 + (-1)^2}$

$= 6.71 \left(= \sqrt{45} = 3\sqrt{5}\right)$ ***A1***

[7 marks]

continued…

Question 12 continued

(d) let B' be the image of B

METHOD 1

$\overrightarrow{OB'} = \begin{pmatrix} -3 \\ 12 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$ ***(A1)***

recognition that $\mu = 2\lambda\,(= -6)$ OR $|BC| = |CB'|$ (may be seen in a diagram) ***(M1)***

$\overrightarrow{OB'} = \begin{pmatrix} -3 \\ 0 \\ 8 \end{pmatrix}$

so coordinates are $B'(-3, 0, 8)$ ***A1***

METHOD 2

$\overrightarrow{BC} = \overrightarrow{CB'} = \begin{pmatrix} 0 \\ -6 \\ 3 \end{pmatrix}$ ***(A1)***

Note: This may come from $\overrightarrow{BC} = -3\sqrt{5}\boldsymbol{n}$ using the unit normal vector $\boldsymbol{n} = \frac{1}{\sqrt{5}}\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$

$\overrightarrow{OB'} = \overrightarrow{OC} + \overrightarrow{CB'}$ OR $\overrightarrow{OB'} = \overrightarrow{OB} + 2\overrightarrow{BC}$ OR $\overrightarrow{OB'} = 2\overrightarrow{OC} - \overrightarrow{OB}$ ***(M1)***

$= \begin{pmatrix} -3 \\ 6 \\ 5 \end{pmatrix} + \begin{pmatrix} 0 \\ -6 \\ 3 \end{pmatrix}$ OR $\begin{pmatrix} -3 \\ 12 \\ 2 \end{pmatrix} + 2\begin{pmatrix} 0 \\ -6 \\ 3 \end{pmatrix}$ OR $\begin{pmatrix} -6 \\ 12 \\ 10 \end{pmatrix} - \begin{pmatrix} -3 \\ 12 \\ 2 \end{pmatrix}$

$\overrightarrow{OB'} = \begin{pmatrix} -3 \\ 0 \\ 8 \end{pmatrix}$ (so coordinates are $B'(-3, 0, 8)$) ***A1***

[3 marks]

Total [18 marks]